

Homoclinic Invariants of Ergodic Actions

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Abstract

We consider a family of homoclinic groups and Gordin's type invariants of measure-preserving actions, state their connections with factors, full groups, ranks, rigidity, multiple mixing and realize such invariants in the class of Gaussian and Poisson suspensions.

1 Introduction

In a topological group G there is the correspondence between an element T and its homoclinic group:

$$H(T) = \{S : T^{-n}ST^n \rightarrow I, n \rightarrow \infty\},$$

where I is the neutral element. We consider the case where G is the group of all automorphisms of a Lebesgue space. It is not hard to see that all Bernoulli actions have ergodic homoclinic groups (the same is probably true for all K-automorphisms). King, answering Gordin's question, built in [2] a zero entropy transformation T of a probability space with ergodic $H(T)$, we say a *transformation T with Gordin's G -property*. This invariant implies the mixing [1], furthermore, it implies the mixing of all orders [8].

All mixing Gaussian and Poisson suspensions (see [3] for definitions), have G -property (the proof for Poisson suspensions see in [8], as for Gaussian actions, it is an exercise).

We know that all group actions without multiple mixing property and the horocycle flows do not possess Gordin's property (the latter follows from Ratner's results [6]).

Let us define a family of homoclinic nature groups. The weak homoclinic

group is defined as

$$WH(T) = \{S \in G : \frac{1}{N} \sum_{n=1}^N T^{-n}ST^n \rightarrow I, N \rightarrow \infty\}.$$

For P , an infinite subset of integers, we define the group

$$H_P(T) = \{S \in G : T^{-n}ST^n \rightarrow I, n \in P, n \rightarrow \infty\}.$$

One presumes here the strong operator convergence, associating the transformations with corresponding operators in $L_2(X, \mu)$.

2 Results

A transformation T is prime by definition as it possesses only trivial invariant σ -algebras. Recall that the full group $[S]$ of the automorphism S is defined as the collection of all automorphisms R of the Lebesgue space (X, μ) such that for all $x \in X$ one has $R(x) = S^{n(x)}$. The full group $[\{S_g\}]$ of a family $\{S_g\}$ is the group generated by all groups $[S_g]$.

Theorem 1. *If a transformation T is prime, then either the group $H_P(T)$ is ergodic or $H_P(T) = \{I\}$.*

For all P the group $H_P(T)$ is full : $H_P(T) = [H_P(T)]$.

Corollary. *If the group $WH(T)$ ($H(T), H_P(T)$) is ergodic, then it includes representatives of all conjugacy classes of ergodic transformations. King's homoclinic group from [2] contains isomorphic copies of all ergodic transformations.*

Theorem 2. *Weakly mixing Gaussian and Poisson suspensions have ergodic weak homoclinic group.*

Let T be an automorphism of a standard Lebesgue space (X, μ) , $\mu(X) = 1$. Let for a number $\beta > 0$ there exist a sequence ξ_j of partitions of X in the form

$$\xi_j = \{B_j, TB_j, T^2B_j, \dots, T^{h_j-1}B_j, C_j^1, \dots, C_j^{m_j} \dots\},$$

and any measurable set can be approximated by ξ_j -measurable ones as $j \rightarrow \infty$, and $\mu(U_j) \rightarrow \beta$, where $U_j = \sqcup_{0 \leq k < h_j} T^k B_j$. The local rank $\beta(T)$ is defined

as maximal number β for which the automorphism T possesses a corresponding sequence of approximating partitions. An automorphism T is said to be of rank 1, if $\beta(T) = 1$.

Theorem 3. *If for an automorphism T of rank 1 the group $WH(T)$ is infinite, then T is rigid: for some sequence $n_i \rightarrow \infty$ it holds $T^{n_i} \rightarrow I$.*

If a mixing transformation R is of rank 1, then $H(R) = WH(R) = \{I\}$.

If an automorphism T is ergodic, $\beta(T) = \beta > 0$ and the group $WH(T)$ is infinite, then there is a sequence $n_i \rightarrow \infty$ such that $T^{n_i} \rightarrow \beta I + (1 - \beta)M$ for some Markov operator M (we say: T is partially rigid).

Theorems 2,3 imply a generalization of Parreau-Roy's result: rank-one Poisson suspension must be rigid [5] (for now there are no examples of such kind).

Theorem 4. *For any two infinite families of integers there are some subsets Q, P of these families, respectively, such that for some weakly mixing Poisson (Gaussian) suspensions T, T' the groups $H_P(T), H_Q(T')$ are ergodic and the groups $H_P(T'), H_Q(T)$ are trivial.*

The homoclinic approach gives new proofs of the multiple mixing for Poisson suspensions (Roy [7]) and Gaussian actions (Leonov [4]) as well as the proof of weak multiple mixing for weakly mixing Gaussian and Poisson suspensions.

Theorem 5. *Suppose that an automorphism T satisfies the properties $H_P(T)$ is ergodic, and for sequences $m_i^1, \dots, m_i^k \in P$, $m_i^1, m_i^k \rightarrow \infty$ the convergence*

$$\mu(T^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} B_k) \rightarrow \mu(B_1) \dots \mu(B_k)$$

holds for any measurable sets B_1, \dots, B_k .

Then for any measurable sets B, B_1, \dots, B_k we have

$$\mu(B \cap T^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} B_k) \rightarrow \mu(B) \mu(B_1) \dots \mu(B_k).$$

3 Proofs

Proof of Theorem 1. It is not hard to see that the algebra of the fixed sets with respect to a homoclinic group $H_P(T)$ is a factor of T . Indeed, if $T^{-n_i}ST^{n_i} \rightarrow I$, then

$$T^{-n_i-1}ST^{n_i+1} \rightarrow I, \quad T^{-n_i}T^{-1}STT^{n_i} \rightarrow I.$$

Thus, from $A = T^{-1}STA$ we obtain $TA = STA$, hence, we have the first part of Theorem 1.

The second part of Theorem 1 is an exercise as well.

Proof of Theorem 2. Let a weakly mixing Poisson suspension T_* be induced by infinite transformation T . The latter has the ergodic weak homoclinic group. It follows from the fact that all finite measure supports infinite transformations F are weakly homoclinic:

$$\frac{1}{N} \sum_{n=1}^N T^{-n}FT^n \rightarrow I, \quad N \rightarrow \infty.$$

Analogous arguments for mixing transformations see in [8].

For Gaussian actions our method is similar: we use the fact that the group FO of all "finite dimension" orthogonal operators are dense in the group of all orthogonal operators on l_2 . The term "finite dimension" orthogonal operator means that this operator is in the form $U \oplus I$, where U acts on a finite dimension space. For any weakly mixing Gaussian transformation T the Gaussian image of the group FO will be an ergodic subgroup of the group $WH(T)$.

The proof of Theorem 3 requires the methods of [9].

Proof of Theorem 4. It is an exercise to construct infinite transformations T mixing along some subset P and rigid along some subset Q . Then we apply the corresponding Poisson and Gaussian constructions.

Proof of Theorem 5. Let for a measure ν we have

$$\mu(B \cap T^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} B_k) \rightarrow \nu(B \times B_1 \times \dots \times B_k)$$

for any measurable sets B, B_1, \dots, B_k .

The projection of the measure ν into the cube $X_1 \times \dots \times X_k$, where X denotes the phase space of our transformation, is μ^k .

The measure ν is invariant with respect to $S \times Id \times Id \dots \times Id$ for all $S \in H_P(T)$. Indeed,

$$\begin{aligned} & \mu(B \cap T^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} B_k) \\ &= \mu(SB \cap ST^{m_i^1} B_1 \cap \dots \cap ST^{m_i^k} B_k) \\ &= \mu(SB \cap T^{m_i^1} T^{-m_i^1} ST^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} T^{-m_i^k} ST^{m_i^k} B_k). \end{aligned}$$

But

$$\mu(SB \cap T^{m_i^1} T^{-m_i^1} ST^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} T^{-m_i^k} ST^{m_i^k} B_k) - \mu(SB \cap T^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} B_k) \rightarrow 0.$$

From this we get

$$\mu(B \cap T^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} B_k) \rightarrow \nu(SB \times B_1 \times \dots \times B_k),$$

hence, for all $S \in H_P(T)$

$$\nu(B \times B_1 \times \dots \times B_k) = \nu(SB \times B_1 \times \dots \times B_k).$$

The ergodicity of the group $H_P(T)$ implies now

$$\nu = \mu \times \mu^k = \mu^{k+1}.$$

Thus,

$$\mu(B \cap T^{m_i^1} B_1 \cap \dots \cap T^{m_i^k} B_k) \rightarrow \mu(B) \mu(B_1) \dots \mu(B_k).$$

4 Questions

Let T be weakly mixing rank-one transformation. What can we say about homoclinic groups $WH(T)$, $H_P(T)$?

We note that

$$\bigcup_{P, d(P)=1} H_P(T) \subset WH(T),$$

where $d(P)$ denotes the density of the set P .

Could $H_P(T)$ be ergodic for mixing rank-one transformation T , or, more generally, for a transformation with minimal self-joinings?

From Ageev's results we know that for generic actions of the group $(S, T : T^{-1}S^2T = S)$ the generator S is rank-one. In connection with the above question it is naturally to ask: could such T be a rank-one automorphism? For all known models the corresponding element T is out of rank one. In generic case for some infinite set $P \subset \{2, 4, 8, \dots\}$ the group $H_P(T)$ is ergodic.

Only for transformations R with discrete spectrum we know that $H_P(R) = \{I\}$ for all infinite sets P .

Are there weakly mixing transformations with the same property?

References

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